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# Technical Comments

## Comment on "Robustness of Solutions to a Benchmark Control Problem"

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REFERENCE 1 is a comparative study of a number of solutions to a common benchmark problem. Stochastic robustness analysis is used to assess the performance and stability robustness of a number of controllers in the presence of plant model error. This error is assumed to take the form of random parameter variations with specified probability density functions. Monte Carlo methods are used to estimate the probability of instability $P_I$.

It is found in Ref. 1 that the gain and phase margins (GM and PM) of the nominal systems are not good predictors of $P_I$. Although increasing parameter uncertainty usually increases $P_I$, there are no consistent trends with GM and PM. This important and surprising result contradicts extensive experience that GM and PM are reliable measures of stability robustness for single-input/single-output (SISO) systems. It is important to understand the extent to which the result is applicable outside the immediate context of the benchmark problem. The purpose of this comment is to supplement the authors' discussion of this point.

The authors attribute their findings mainly to the fact that the shape of the Nyquist plot of the open-loop transfer function varies in a complicated way as a function of the plant uncertainties. The implication is that placing bounds on the closeness of approach of the Nyquist plot to the critical point at the two cross-over frequencies provides little assurance that it will not cross the critical point at some other frequency. Of course this is true; but this Note also argues that one would not expect GM and PM to be reliable robustness measures for a class of systems that includes the example used in Ref. 1.

Let the open-loop transfer function be $L(s)$, and let the $s$-plane contour on which $L(s)$ is evaluated, in applying the Nyquist stability criterion, be $C$. Normally $C$ runs along the imaginary $s$ axis with closure to the right at infinity. GM and PM are measures of how much $L(s)$ can change on $C$ before the encirclement count, and hence the number of closed-loop poles inside $C$ is altered. Values of GM and PM should not be quoted blindly unless there is some confidence that the magnitude and phase errors on $C$ will actually be smaller than the margins over the relevant frequency range. If $L(s)$ has poles or zeros near to $C$, then even small variations in their locations due to model error can give rise to an error in $L(s)$ on $C$ that is much larger than that covered by the margins. A simple example is benchmark problem 1 (BP-1) of Ref. 1. The plant model can be written as $G(s) = p^2/2s^2(s^2+p^2)$ where $1 \leq p \leq 2$ and the nominal value is $\sqrt{2}$. The loop transfer function is given by $L(s) = G(s)K(s)$, where $K(s)$ is the controller. Even if the portion of $C$ from $s = j$ to $s = 2j$ is indented infinitesimally to the left or right so that $L(s)$ is defined over all of the contour for all values of $p$, the magnitude and phase errors on and near this part of the contour, due to the displacement of poles from their nominal positions, are too large for meaningful conclusions to be drawn about the stability robustness from GM and PM. Similar remarks can be made about any SISO plant that has nominal poles (or zeros) near the imaginary axis. Nevertheless, meaningful statements about stability robustness can be made in such cases if $C$ is indented to the left to avoid the region of the $s$ plane where the anticipated model error is large.[2] Margins computed on the modified contour are reliable, since the anticipated error on the contour is small. The implication, in designing for stability robustness, is that nominal closed-loop poles should be placed to the left of the modified contour. Note that the movement of a closed-loop pole to the right of the indented contour does not necessarily mean that it will also move into the right-hand $s$ plane. To minimize conservatism the contour should be indented no more than necessary to reduce the anticipated error on the indentation to an acceptable level.

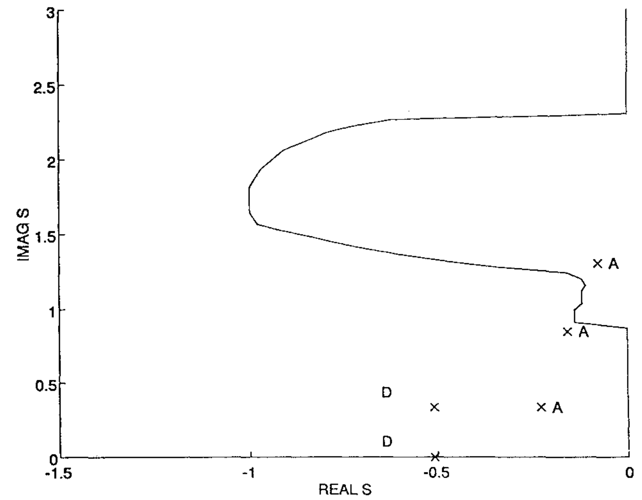


**Fig. 1 Nominal closed-loop poles for designs A and D from Ref. 1. Also shown is indentation of Nyquist contour to the left to avoid area in *s* plane of high expected model error.**

Consider problem BP-1 from Ref. 1 for which, fortunately, we have an exact characterization of the error. Figure 1 illustrates a modified contour for the plant model of BP-1. The indentation was chosen so that, between it and the imaginary $s$ axis, the gain error exceeds 6 dB and the phase error exceeds 30 deg. Also shown in Fig. 1 are nominal closed-loop poles for designs A and D from Ref. 1. Design A has poles to the right of the indentation and, unsurprisingly, also has a relatively high $P_I$ of 0.160. In contrast, all of the poles of design D lie to the left of the contour. The Nyquist plot shows that there are two sets of margins. Both phase margins are in excess of 30 deg. One gain margin is bigger than 6 dB, the other being about 4.4 dB (against gain decrease). In view of the conservatism mentioned earlier, it comes as no surprise to note that $P_I$ for design $D$ was 0.000.

In summary, traditional gain and phase margins may be misleading as indicators of robustness for SISO plant models having poles or zeros near the imaginary axis, because small movements of the poles and zeros can give rise to very large magnitude and phase errors on nearby portions of the imaginary axis. Reliable margins can be obtained by indenting the contour $C$ to the left, avoiding regions of high expected model error.

## References

[1]Stengel, R. F., and Marrison, C. I., "Robustness of Solutions to a Benchmark Control Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, 1992, pp. 1060–1067.

[2]Gribble, J. J., "Individual Channel Design in the $s$-Plane," *Systems-Analysis-Modelling-Simulation*, Vol. 15, No. 2, 1994, pp. 115–134.

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